

ON PERMUTIPLES HAVING A FIXED SET OF DIGITS

Benjamin V. Holt

Department of Mathematics, Humboldt State University, Arcata, CA 95521, USA
bvh6@humboldt.edu**Abstract**

A permutiple is the product of a digit preserving multiplication, that is, a number which is an integer multiple of some permutation of its digits. Certain permutiple problems, particularly transposable, cyclic, and, more recently, palintiple numbers, have been well-studied. In this paper we study the problem of general digit preserving multiplication. We show how the digits and carries of a permutiple are related and utilize these relationships to develop methods for finding new permutiple examples from old. In particular, we shall focus on the problem of finding new permutiples from a known example having the same set of digits.

1. Introduction

A *permutiple* is a natural number with the property of being an integer multiple of some permutation of its digits. Digit permutation problems are nothing new [2, 10] and have been a topic of study of both amateurs and professionals alike [5]. A relatively well-studied example of permutiples includes *palintiple* numbers, also known as reverse multiples [6, 9, 11], which are integer multiples of their digit reversals and include well-known base-10 examples such as $87912 = 4 \cdot 21978$ and $98901 = 9 \cdot 10989$. As noted by Sutcliffe [10] in his seminal paper on palintiple numbers, cyclic digit permutations such as $714285 = 5 \cdot 142857$ are also well-studied examples. We also note that 142857 is an example of a *cyclic number*; not only does multiplication by 5 permute the digits, but 2, 3, 4, and 6 also produce cyclic digit permutations.

Permutiples for which the digits are cyclically permuted are relatively well-understood and their description is fairly straightforward in comparison to palintiples. The digits of cyclic permutiples are found in repeating base- b decimal expansions of the form a/p where $a < p$ and p is a prime which does not divide b [2, 5]. On the other hand, palintiples (digit reversing permutations) admit quite a variety of classifications [6, 7] and are not nearly as well-understood. Young [11, 12], building upon the body of work of Sutcliffe [10] and others [1, 8], translates

the palintiple problem into graph-theoretical language by representing an efficient palintiple search method as a tree graph where the possible carries are represented as nodes and the potential digits are associated with the edges. Continuing the work of Young [11, 12], Sloane [9] modified Young's tree graph representation into the *Young graph* which is a visualization of digit-carry palintiple structure. The paper identifies and studies several Young graph isomorphism classes which describe palintiple type. Furthering the work of Sloane [9], Kendrick [6] proves two of Sloane's [9] main conjectures involving Young graph isomorphism classes which describe two well-understood palintiple types. The work of [3, 4] takes a more elementary approach and classifies palintiples according to patterns exhibited by their carries which, as noted by Kendrick [6], seems to coincide with Young graph isomorphism classes with [4] conjecturing that the class of palintiples characterized by the 1089 graph are precisely the collection of *symmetric palintiples* (the carry sequence is palindromic) described in [3]. Kendrick's work [6, 7] reveals the sheer multitude of palintiple types when classified according to Young graph isomorphism.

In this paper we establish some general properties of digit preserving multiplication. We generalize the results for palintiple numbers found in Young [11], Sloane [9], Kendrick [6], and Holt [3] to an arbitrary permutation. Using these results, we develop methods for finding new permutiples from old. In particular, we consider the problem of finding new base- b permutiples with multiplier n having a fixed set of digits from a single known example. Moreover, we find a condition under which our methods give us all permutiples of a particular base and multiplier having the same digits as a known example.

2. Permutiple Digits and Carries

We begin with a definition. We shall use $(d_k, d_{k-1}, \dots, d_0)_b$ to denote the natural number $\sum_{j=0}^k d_j b^j$ where each $0 \leq d_j < b$.

Definition Let n be a natural number and σ be a permutation on $\{0, 1, 2, \dots, k\}$. We say that $(d_k, d_{k-1}, \dots, d_0)_b$ is an (n, b, σ) -*permutiple* provided

$$(d_k, d_{k-1}, \dots, d_1, d_0)_b = n(d_{\sigma(k)}, d_{\sigma(k-1)} \dots, d_{\sigma(1)}, d_{\sigma(0)})_b.$$

Using the language established above, letting $\rho = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$, 87912 is a $(4, 10, \rho)$ -permutiple.

In order to avoid introducing extra digits when multiplying it is assumed that $n < b$. We do note, however, that in order to circumvent overly cumbersome theorem statements, we do allow for leading zeros. Letting ε be the identity permutation, every natural number is a $(1, b, \varepsilon)$ -permutiple. Such trivial examples are ignored so that $n > 1$. Furthermore, $b = 2$ implies that $n = 1$. Therefore, we impose the

additional restriction that $b \neq 2$. Thus, hereafter we assume that n and b are natural numbers such that $1 < n < b$ and $b > 2$.

The following two theorems are an exceedingly straightforward generalization of Theorems 1 and 3 found in [3] for palintiple numbers. A description of single-digit multiplication in general is as follows: let p_j denote the j th digit of the product, c_j the j th carry, and q_j the j th digit of the number being multiplied by n . Then the iterative algorithm for single digit multiplication is

$$\begin{aligned} c_0 &= 0 \\ p_j &= \lambda(nq_j + c_j) \\ c_{j+1} &= [nq_j + c_j - \lambda(nq_j + c_j)] \div b \end{aligned}$$

where λ gives the least non-negative residue modulo b . Since $(p_k, p_{k-1}, \dots, p_0)_b$ is a $k+1$ digit number, $c_{k+1} = 0$. For any (n, b, σ) -permutiple $(d_k, d_{k-1}, \dots, d_0)_b$, $q_j = d_{\sigma(j)}$ so that $d_j = p_j = \lambda(nd_{\sigma(j)} + c_j)$. Hence,

Theorem 1. *Let $(d_k, d_{k-1}, \dots, d_0)_b$ be an (n, b, σ) -permutiple and let c_j be the j th carry. Then*

$$bc_{j+1} - c_j = nd_{\sigma(j)} - d_j$$

for $0 \leq j \leq k$.

As is the case for palintiples, the following shows that the carries of any permutiple are less than the multiplier.

Theorem 2. *Let $(d_k, d_{k-1}, \dots, d_0)_b$ be an (n, b, σ) -permutiple and let c_j be the j th carry. Then $c_j \leq n-1$ for all $0 \leq j \leq k$.*

Proof. The proof will proceed by induction. $c_0 = 0 \leq n-1$. Now suppose $c_j \leq n-1$. For a contradiction suppose $c_{j+1} \geq n$. Then Theorem 1 implies $bc_{j+1} - c_j + d_j = nd_{\sigma(j)}$. By our inductive hypothesis we have $bn - (n-1) = (b-1)n + 1 \leq nd_{\sigma(j)}$. Therefore $d_{\sigma(j)} > b-1$ which is a contradiction. \square

The following is a converse to Theorem 1.

Theorem 3. *Suppose $bc_{j+1} - c_j = nd_{\sigma(j)} - d_j$ for all $0 \leq j \leq k$ where (d_k, \dots, d_0) is a $k+1$ -tuple of base- b digits and (c_k, \dots, c_0) is a $k+1$ -tuple of base- n digits such that $c_0 = 0$. Then $(d_k, \dots, d_0)_b$ is an (n, b, σ) -permutiple with carries c_k, \dots, c_0 .*

Proof.

$$\sum_{j=0}^k (nd_{\sigma(j)} - d_j)b^j = \sum_{j=0}^k (bc_{j+1} - c_j)b^j = 0$$

so that $(d_k, \dots, d_0)_b$ is an (n, b, σ) -permutiple. Letting $(\hat{c}_k, \dots, \hat{c}_0)$ be the carries, an application of Theorem 1 and a simple induction argument establish that $\hat{c}_j = c_j$ for all $0 \leq j \leq k$. \square

Letting ψ be the $k+1$ -cycle $(0, 1, 2, \dots, k)$, it is convenient to write the relations between the digits and the carries found in Theorems 1 and 3 in matrix form:

$$(bP_\psi - I)\mathbf{c} = (nP_\sigma - I)\mathbf{d} \quad (1)$$

where I the identity matrix, P_ψ and P_σ are permutation matrices, and \mathbf{c} and \mathbf{d} are column vectors respectively containing the carries and digits. We note that these matrices are indexed from 0 to k rather than from 1.

We note that we will extensively use the fact that by Theorems 1 and 3 the number $(d_k, \dots, d_0)_b$ is an (n, b, σ) -permutiple with carries $c_k, \dots, c_1, c_0 = 0$ if and only if Equation 1 holds.

Multiplying both sides of Equation 1 by $\sum_{\ell=0}^{|\sigma|-1} (nP_\sigma)^\ell$, we can express the digits in terms of the carries as $\mathbf{d} = \frac{1}{n^{|\sigma|-1}} \sum_{\ell=0}^{|\sigma|-1} (n^\ell P_{\sigma^\ell})(bP_\psi - I)\mathbf{c}$. Similarly, multiplying both sides by $\sum_{\ell=0}^k (bP_\psi)^\ell$, we can likewise express the carries in terms of the digits. $\mathbf{c} = \frac{1}{b^{k+1}-1} \sum_{\ell=0}^k (b^\ell P_{\psi^\ell})(nP_\sigma - I)\mathbf{d}$. In component form we have

Theorem 4. *Let $(d_k, d_{k-1}, \dots, d_0)_b$ be an (n, b, σ) -permutiple and let c_j be the j th carry. Then*

$$d_j = \frac{1}{n^{|\sigma|-1}} \sum_{\ell=0}^{|\sigma|-1} (bc_{\psi\sigma^\ell(j)} - c_{\sigma^\ell(j)})n^\ell$$

and

$$c_j = \frac{1}{b^{k+1}-1} \sum_{\ell=0}^k (nd_{\sigma\psi^\ell(j)} - d_{\psi^\ell(j)})b^\ell$$

for all $0 \leq j \leq k$.

Remark We direct the reader's attention to the symmetry between the above equations expressing the digits in terms of the carries and vice versa. The above also generalizes the relationship between palintiple numbers and their carries found in [3, 11, 9].

The next theorem generalizes Theorem 5 of [3] which places restrictions on the possible values of n , b , and the order of σ .

Theorem 5. *For any nontrivial (n, b, σ) -permutiple, $|\sigma| \geq \frac{(n+1)\gcd(n,b)}{b} + 1$.*

Proof. Let $(d_k, d_{k-1}, \dots, d_0)_b$ be an (n, b, σ) -permutiple $(d_k, d_{k-1}, \dots, d_0)_b$ where c_j is the j th carry. For $j = 0$, Theorem 4 gives $(n^{|\sigma|-1} - 1)d_0 = b \left(\sum_{\ell=0}^{|\sigma|-1} c_{\psi\sigma^\ell(0)} n^\ell \right) - n \left(\sum_{\ell=1}^{|\sigma|-1} c_{\sigma^\ell(0)} n^{\ell-1} \right)$. Thus $\gcd(n, b)$ divides d_0 since n and $n^{|\sigma|-1} - 1$ are relatively prime. Theorem 2 then gives us $\gcd(n, b) \leq d_0 \leq \frac{b}{n^{|\sigma|-1}} \sum_{\ell=0}^{|\sigma|-1} c_{\psi\sigma^\ell(j)} \leq \frac{b(n-1)(|\sigma|-1)}{n^{|\sigma|-1}} \leq \frac{b(n-1)(|\sigma|-1)}{n^2-1} \leq \frac{b(|\sigma|-1)}{n+1}$. \square

3. New Permutiples from Old

We shall now consider the problem of finding new permutiples from known examples. The approach taken here, as stated in the beginning, will be to restrict our attention to finding new permutiples having the same digits as our known example. The ultimate aim of our effort is to answer the question of whether or not all permutiples having the same digits can be found from a single example. And if not, are there conditions under which it is possible? The following results give us some methods for constructing new permutiples from old.

Theorem 6. *Let $(d_k, \dots, d_0)_b$ be an (n, b, σ) -permutiple with carries c_k, \dots, c_0 and let μ be a permutation such that $c_{\mu(0)} = 0$. Then $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$ is an $(n, b, \pi^{-1}\sigma\pi)$ -permutiple with carries $c_{\mu(k)}, c_{\mu(k-1)}, \dots, c_{\mu(0)}$ if and only if $P_\pi(bP_\psi - I)\mathbf{c} = (bP_\psi - I)P_\mu\mathbf{c}$.*

Proof. By our hypothesis, Equation 1 is satisfied. Thus, if $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$ is an $(n, b, \pi^{-1}\sigma\pi)$ -permutiple with carries $c_{\mu(k)}, c_{\mu(k-1)}, \dots, c_{\mu(0)}$, then

$$P_\pi(bP_\psi - I)\mathbf{c} = P_\pi(nP_\sigma - I)\mathbf{d} = (nP_{\pi^{-1}\sigma\pi} - I)P_\pi\mathbf{d} = (bP_\psi - I)P_\mu\mathbf{c}.$$

Conversely, if $P_\pi(bP_\psi - I)\mathbf{c} = (bP_\psi - I)P_\mu\mathbf{c}$, then

$$(nP_{\pi^{-1}\sigma\pi} - I)P_\pi\mathbf{d} = P_\pi(nP_\sigma - I)\mathbf{d} = P_\pi(bP_\psi - I)\mathbf{c} = (bP_\psi - I)P_\mu\mathbf{c}.$$

□

Corollary 7. *Let $(d_k, d_{k-1}, \dots, d_0)_b$ be an (n, b, σ) -permutiple with carries c_k, c_{k-1}, \dots, c_0 . If $c_j = 0$, then $(d_{\psi^j(k)}, d_{\psi^j(k-1)}, \dots, d_{\psi^j(1)}, d_{\psi^j(0)})_b$ is an $(n, b, \psi^{-j}\sigma\psi^j)$ -permutiple with carries $c_{\psi^j(k)}, c_{\psi^j(k-1)}, \dots, c_{\psi^j(1)}, c_{\psi^j(0)} = c_j = 0$.*

Remark We note that the above corollary follows either from Theorem 4 by setting $c_j = 0$, or from Theorem 6 by setting $\pi = \mu = \psi^j$.

Example Consider the $(4, 10, \rho)$ -permutiple $(8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$. Performing routine multiplication see that the carries are $(c_4, c_3, c_2, c_1, c_0) = (0, 3, 3, 3, 0)$. Not surprisingly, applying Corollary 7 to $j = 0$ yields the original permutiple. However, for the case of $j = 4$, we see that $(7, 9, 1, 2, 8)_{10} = 4 \cdot (1, 9, 7, 8, 2)_{10}$ with carries $(c_{\psi^4(4)}, c_{\psi^4(3)}, c_{\psi^4(2)}, c_{\psi^4(1)}, c_{\psi^4(0)}) = (c_3, c_2, c_1, c_0, c_4) = (3, 3, 3, 0, 0)$.

Applying Corollary 7 more generally, if $(d_k, d_{k-1}, \dots, d_0)_b$ is any (n, b) -palintiple such that $c_k = 0$ (this includes all symmetric, doubly-derived, and doubly-reverse-derived palintiples [4]), then $(d_{k-1}, d_{k-2}, \dots, d_0, d_k)_b$ is an $(n, b, \psi^{-k}\rho\psi^k)$ -permutiple.

Setting μ to the identity permutation in Theorem 6, we obtain another useful corollary.

Corollary 8. *Let $(d_k, d_{k-1}, \dots, d_0)_b$ be an (n, b, σ) -permutiple with carries c_k, c_{k-1}, \dots, c_0 . Then $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$ is an $(n, b, \pi^{-1}\sigma\pi)$ -permutiple with carries c_k, c_{k-1}, \dots, c_0 if and only if $P_\pi(bP_\psi - I)\mathbf{c} = (bP_\psi - I)\mathbf{c}$.*

Example Consider again the base-10 palintiple $(8, 7, 9, 1, 2)_{10} = 4 \cdot (2, 1, 9, 7, 8)_{10}$ with carries $(c_4, c_3, c_2, c_1, c_0) = (0, 3, 3, 3, 0)$. With the above in mind, we calculate

$$(10P_\psi - I)\mathbf{c} = \left(10 \cdot \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 30 \\ 27 \\ 27 \\ -3 \\ 0 \end{bmatrix}.$$

Since the above column vector is unchanged by P_π where π is the transposition $(1, 2)$, we see by Corollary 8 that $(8, 7, 1, 9, 2)_{10}$ is a $(4, 10, (2, 3)\rho(2, 3))$ -permutiple with carries $(c_4, c_3, c_2, c_1, c_0) = (0, 3, 3, 3, 0)$ which may be also be confirmed by simple arithmetic.

Performing the same calculation as above, the $(4, 10, \psi^{-4}\rho\psi^4)$ -permutiple $(7, 9, 1, 2, 8)_{10} = 4 \cdot (1, 9, 7, 8, 2)_{10}$ from the previous example yields via Corollary 8 the $(4, 10, ((2, 3)\psi^{-4}\rho\psi^4(2, 3))$ -permutiple $(7, 1, 9, 2, 8) = 4 \cdot (1, 7, 9, 8, 2)_{10}$ with carries $(c_4, c_3, c_2, c_1, c_0) = (3, 3, 3, 0, 0)$. We note that we arrive at the same result by applying Corollary 7 to the $(4, 10, (1, 2)\rho(1, 2))$ -permutiple $(8, 7, 1, 9, 2)_{10}$ which is affirmed by the fact that $(1, 2)\psi^4 = \psi^4(2, 3)$.

At this point a several questions naturally present themselves. Can all permutiples having a certain set of digits be found by repeated use of Theorem 6 and its corollaries? One does not have to look far to see that the answer is no. If we consider the example $(7, 8, 9, 1, 2)_{10} = 4 \cdot (1, 9, 7, 2, 8)_{10}$ with carries $(c_4, c_3, c_2, c_1, c_0) = (3, 2, 1, 3, 0)$ we see that the results obtained thus far do not account for this example since the carries are different.

Another question is if we have an (n, b, σ) -permutiple $(d_k, \dots, d_0)_b$ with carries c_k, \dots, c_0 and an (n, b, τ) -permutiple $(d_{\pi(k)}, \dots, d_{\pi(0)})_b$ with permuted carries $c_{\mu(k)}, c_{\mu(k-1)}, \dots, c_{\mu(0)}$, must $\tau = \pi^{-1}\sigma\pi$? Again, with a little effort, we can find an example which shows this is not always the case. Consider $(4, 3, 5, 1, 2)_6 = 2 \cdot (2, 1, 5, 3, 4)_6$, a $(2, 6, \sigma)$ -permutiple, and $(2, 5, 1, 3, 4)_6 = 2 \cdot (1, 2, 3, 4, 5)_6$, a $(2, 6, \tau)$ -permutiple, both with the same carry vector $(0, 1, 1, 1, 0)$. Now $\sigma = (0, 4)(1, 3)$, $\pi = (0, 4)(3, 2, 1)$, and $\tau = (0, 3, 4, 2, 1)$, but π but $\tau \neq \pi^{-1}\sigma\pi$.

Thus, it is clear that our results so far do not account for every possibility. Therefore, we shall require some additional machinery in order to find every permutiple with the same digits as our known example. Again, for the purpose of less cumbersome theorem statements, we shall henceforth assume that $(d_k, \dots, d_0)_b$ is an (n, b, σ) -permutiple. We begin with a definition, motivated by the above, which will help us to organize and classify our new examples.

Definition We say that the (n, b, τ_1) -permutiple $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$ and the (n, b, τ_2) -permutiple $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$ are *conjugate* if $\pi_1 \tau_1 \pi_1^{-1} = \pi_2 \tau_2 \pi_2^{-1}$.

Clearly permutiple conjugacy defines an equivalence relation on the collection of all base- b permutiples with multiplier n having the same digits. From this fact, we also establish some additional terminology. For any two (n, b, τ_1) and (n, b, τ_2) -permutiples of the same conjugacy class, $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$ and $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$, we shall refer to the common permutation $\beta = \pi_1 \tau_1 \pi_1^{-1} = \pi_2 \tau_2 \pi_2^{-1}$ as the *base permutation* of the class. We must emphasize that the base permutation of a conjugacy class might not necessarily be a digit permutation itself.

Our next result tells us that two permutiples in the same conjugacy class both have the same set of carries.

Theorem 9. *Let $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$ and $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$ be respectively (n, b, τ_1) and (n, b, τ_2) -permutiples from the same conjugacy class and that their carries are respectively given by c_k, c_{k-1}, \dots, c_0 and $\hat{c}_k, \hat{c}_{k-1}, \dots, \hat{c}_0$. Then $\hat{c}_j = c_{\pi_1^{-1} \pi_2(j)}$ for all $0 \leq j \leq k$.*

Proof. By assumption we have both that $(nP_{\tau_1} - I)P_{\pi_1} \mathbf{d} = (bP_\psi - I)\mathbf{c}$ and that $(nP_{\tau_2} - I)P_{\pi_2} \mathbf{d} = (bP_\psi - I)\hat{\mathbf{c}}$. Then $P_{\pi_1}(nP_{\pi_1 \tau_1 \pi_1^{-1}} - I)\mathbf{d} = (bP_\psi - I)\mathbf{c}$ and $P_{\pi_2}(nP_{\pi_2 \tau_2 \pi_2^{-1}} - I)\mathbf{d} = (bP_\psi - I)\hat{\mathbf{c}}$. Since both permutiples are conjugate we have that $P_{\pi_1 \tau_1 \pi_1^{-1}} = P_{\pi_2 \tau_2 \pi_2^{-1}}$. It follows that $P_{\pi_1^{-1}}(bP_\psi - I)\mathbf{c} = P_{\pi_2^{-1}}(bP_\psi - I)\hat{\mathbf{c}}$. Reducing modulo b we have $P_{\pi_1^{-1}} \mathbf{c} \equiv P_{\pi_2^{-1}} \hat{\mathbf{c}} \pmod{b}$, or $P_{\pi_1^{-1} \pi_2} \mathbf{c} \equiv \hat{\mathbf{c}} \pmod{b}$. Theorem 2 then implies that $\hat{\mathbf{c}} = P_{\pi_1^{-1} \pi_2} \mathbf{c}$. \square

The above theorem proves that in Theorem 6, $P_\mu \mathbf{c}$ must equal $P_\pi \mathbf{c}$. Moreover, it also gives us the following.

Theorem 10. *Let $p = (d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$ be an (n, b, τ_1) -permutiple with carries c_k, c_{k-1}, \dots, c_0 . If $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$ is an (n, b, τ_2) -permutiple from the same conjugacy class as p , then $c_{\psi \pi_1^{-1} \pi_2(j)} = c_{\pi_1^{-1} \pi_2 \psi(j)}$ for all $0 \leq j \leq k$.*

Proof. Our second assumption in matrix form is $(nP_{\tau_1} - I)P_{\pi_1} \mathbf{d} = (bP_\psi - I)\mathbf{c}$ and by Theorem 9 our third is $(nP_{\tau_2} - I)P_{\pi_2} \mathbf{d} = (bP_\psi - I)P_{\pi_1^{-1} \pi_2} \mathbf{c}$. Using the second equation we have $(nP_{\pi_2 \tau_2 \pi_2^{-1}} - I)\mathbf{d} = P_{\pi_2^{-1}}(nP_{\tau_2} - I)P_{\pi_2} \mathbf{d} = P_{\pi_2^{-1}}(bP_\psi - I)P_{\pi_1^{-1} \pi_2} \mathbf{c}$ which by conjugacy gives $(nP_{\pi_1 \tau_1 \pi_1^{-1}} - I)\mathbf{d} = P_{\pi_2^{-1}}(bP_\psi - I)P_{\pi_1^{-1} \pi_2} \mathbf{c}$. Multiplying by P_{π_1} we then have $(nP_{\tau_1} - I)P_{\pi_1} \mathbf{d} = P_{\pi_1} P_{\pi_2^{-1}}(bP_\psi - I)P_{\pi_1^{-1} \pi_2} \mathbf{c}$ which by the first relation above becomes $(bP_\psi - I)\mathbf{c} = P_{\pi_1^{-1} \pi_2}^{-1}(bP_\psi - I)P_{\pi_1^{-1} \pi_2} \mathbf{c}$. The above reduces to $P_{\pi_1^{-1} \pi_2} P_\psi \mathbf{c} = P_\psi P_{\pi_1^{-1} \pi_2} \mathbf{c}$ which completes the proof. \square

With the above theorem we may now determine suitable permutations π_1 and π_2 within a particular conjugacy class. Letting $\pi = \pi_1^{-1}\pi_2$, we have under the conditions of the above theorem that $P_\pi P_\psi \mathbf{c} = P_\psi P_\pi \mathbf{c}$. Thus, if π is in the centralizer of ψ , then π is simply some power of ψ as the centralizer of ψ has $k+1$ elements. By arguments similar to those leading up to Corollary 7, this power is precisely any ℓ such that $c_\ell = 0$. Thus, can only be $\pi = \psi^\ell$ if all of the non-zero entries of \mathbf{c} are distinct. To account for repeated non-zero entries of \mathbf{c} , we let ξ be a permutation such that P_ξ simultaneously fixes both \mathbf{c} and $P_\psi \mathbf{c}$. Then $P_\pi P_\psi \mathbf{c} = P_\psi P_\pi \mathbf{c}$ becomes $P_\pi P_\xi P_\psi \mathbf{c} = P_\psi P_\pi P_\xi \mathbf{c}$, or $P_{\xi\pi} P_\psi \mathbf{c} = P_\psi P_{\xi\pi} \mathbf{c}$. Summarizing the above, we have the following.

Theorem 11. *Let $p = (d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$ be an (n, b, τ_1) -permutiple with carries c_k, c_{k-1}, \dots, c_0 . If $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$ is an (n, b, τ_2) -permutiple from the same conjugacy class as p . Then $\pi_2 = \pi_1 \xi \psi^\ell$ where $c_\ell = 0$ and ξ is a permutation such that $c_{\psi\xi(j)} = c_{\psi(j)}$ and $c_{\xi(j)} = c_j$ for all $0 \leq j \leq k$.*

We can see how Corollary 8 accounts for repeated entries in the carries of our known (n, b, σ) -permutiple $(d_k, \dots, d_0)_b$. Thus, by the above, we see applying Corollaries 7 and 8 to a particular conjugacy class gives us all the examples within that class.

The above results give us information about how permutiples within a particular conjugacy class are related to one another. We shall now look between conjugacy classes. The converse of Theorem 9 does not hold in general. However, assuming its consequent does yield a useful theorem which gives us a list of base permutation candidates from every conjugacy class with the same set of carries as the original example.

Theorem 12. *Let $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$ and $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$ be respectively (n, b, τ_1) and (n, b, τ_2) -permutiples with carries respectively given by c_k, c_{k-1}, \dots, c_0 and $\hat{c}_k, \hat{c}_{k-1}, \dots, \hat{c}_0$. If $\hat{c}_j = c_{\pi_1^{-1}\pi_2(j)}$ for all $0 \leq j \leq k$, then $nd_{\pi_1\tau_1\pi_1^{-1}(j)} \equiv nd_{\pi_2\tau_2\pi_2^{-1}(j)} \pmod{b}$ for all $0 \leq j \leq k$.*

Proof. Our assumptions in matrix form are $(nP_{\tau_1} - I)P_{\pi_1} \mathbf{d} = (bP_\psi - I)\mathbf{c}$ and $(nP_{\tau_2} - I)P_{\pi_2} \mathbf{d} = (bP_\psi - I)P_{\pi_1^{-1}\pi_2} \mathbf{c}$. Reducing modulo b we have both $(nP_{\tau_1} - I)P_{\pi_1} \mathbf{d} \equiv -\mathbf{c} \pmod{b}$ and $(nP_{\tau_2} - I)P_{\pi_2} \mathbf{d} \equiv -P_{\pi_1^{-1}\pi_2} \mathbf{c} \pmod{b}$. It follows that $P_{\pi_1}(nP_{\pi_1\tau_1\pi_1^{-1}} - I)\mathbf{d} \equiv -\mathbf{c} \pmod{b}$ and $P_{\pi_2}(nP_{\pi_2\tau_2\pi_2^{-1}} - I)\mathbf{d} \equiv -P_{\pi_1^{-1}\pi_2} \mathbf{c} \pmod{b}$ from which we obtain $(nP_{\pi_1\tau_1\pi_1^{-1}} - I)\mathbf{d} \equiv -P_{\pi_1^{-1}} \mathbf{c} \equiv (nP_{\pi_2\tau_2\pi_2^{-1}} - I)\mathbf{d} \pmod{b}$. Thus $nP_{\pi_1\tau_1\pi_1^{-1}} \mathbf{d} \equiv nP_{\pi_2\tau_2\pi_2^{-1}} \mathbf{d} \pmod{b}$. \square

Letting π_1 be the identity and τ_1 be σ in the above theorem, we obtain a result which relates any (n, b, τ) -permutiple with the same set of carries to our known example.

Corollary 13. *Let $(d_k, \dots, d_0)_b$ be an (n, b, σ) -permutiple with carries c_k, \dots, c_0 and let $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$ be an (n, b, τ) -permutiple with carries $c_{\pi(k)}, c_{\pi(k-1)}, \dots, c_{\pi(0)}$. Then $nd_{\sigma(j)} \equiv nd_{\pi\tau\pi^{-1}(j)} \pmod{b}$ for all $0 \leq j \leq k$.*

The above corollary will, under certain conditions, enable us to find all base permutations $\beta = \pi\tau\pi^{-1}$ for every possible conjugacy class.

Our next result gives us conditions for the existence of a bijective correspondence between permutiples, namely, $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b \mapsto (d_{\alpha\pi(k)}, d_{\alpha\pi(k-1)}, \dots, d_{\alpha\pi(0)})_b$.

Theorem 14. *Let $(d_{\pi_1(k)}, d_{\pi_1(k-1)}, \dots, d_{\pi_1(0)})_b$ and $(d_{\pi_2(k)}, d_{\pi_2(k-1)}, \dots, d_{\pi_2(0)})_b$ be respectively (n, b, τ_1) and (n, b, τ_2) -permutiples. If there exists an α such that $(nP_{\tau_2} - I)P_\alpha \mathbf{d} = (nP_{\tau_1} - I)\mathbf{d}$, then $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$ is an $(n, b, \pi^{-1}\tau_1\pi)$ -permutiple if and only if $(d_{\alpha\pi(k)}, d_{\alpha\pi(k-1)}, \dots, d_{\alpha\pi(0)})_b$ is an $(n, b, \pi^{-1}\tau_2\pi)$ -permutiple.*

Proof. Suppose $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$ is an $(n, b, \pi^{-1}\tau_1\pi)$ -permutiple with carries c_k, c_{k-1}, \dots, c_0 . Then $(nP_{\pi^{-1}\tau_1\pi} - I)P_\pi \mathbf{d} = (bP_\psi - I)\mathbf{c}$. Now by the theorem hypothesis we have $(nP_{\pi^{-1}\tau_1\pi} - I)P_\pi \mathbf{d} = P_\pi(nP_{\tau_1} - I)\mathbf{d} = P_\pi(nP_{\tau_2} - I)P_\alpha \mathbf{d} = (nP_{\pi^{-1}\tau_2\pi} - I)P_{\alpha\pi} \mathbf{d}$ so that $(nP_{\pi^{-1}\tau_2\pi} - I)P_{\alpha\pi} \mathbf{d} = (bP_\psi - I)\mathbf{c}$. By Theorem 3, the forward implication holds. The reverse implication follows in similar fashion. \square

Thus, the above gives us a bijection between conjugacy classes provided $\pi_1\tau_1\pi_1^{-1} \neq \pi_2\tau_2\pi_2^{-1}$. Also, the reader should note that the carries of $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$ and $(d_{\alpha\pi(k)}, d_{\alpha\pi(k-1)}, \dots, d_{\alpha\pi(0)})_b$ must be the same according to the above argument.

At this point the big question is whether or not the results given thus far can give us all the examples we seek. If we recall our initial examples which motivated our conjugacy class definition, we can see that, in general the answer is no. However, there is a condition which guarantees that we have found all of the desired examples. The next theorem tells us that if n divides b , then all permutiples having the same digits as our known example have the same set of carries as the known example.

Theorem 15. *Let $(d_k, \dots, d_0)_b$ be an (n, b, σ) -permutiple with carries c_k, \dots, c_0 and let $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$ be an (n, b, τ) -permutiple with carries $\hat{c}_k, \hat{c}_{k-1}, \dots, \hat{c}_0$. If n divides b , then $\hat{c}_j = c_{\pi(j)}$ for all $0 \leq j \leq k$.*

Proof. By Equation 1, we have both $(bP_\psi - I)\mathbf{c} = (nP_\sigma - I)\mathbf{d}$ and $(bP_\psi - I)\hat{\mathbf{c}} = (nP_\tau - I)P_\pi \mathbf{d}$. Since n divides b , it follows that $\mathbf{c} \equiv \mathbf{d} \pmod{n}$ and $\hat{\mathbf{c}} \equiv P_\pi \mathbf{d} \pmod{n}$. Thus, $\hat{\mathbf{c}} \equiv P_\pi \mathbf{c} \pmod{n}$. By Theorem 2 it follows that $\hat{\mathbf{c}} = P_\pi \mathbf{c}$, which establishes the theorem. \square

Thus, when n divides b , Corollary 13 enables us to find every possible base permutation $\beta = \pi\tau\pi^{-1}$. Using Theorem 11 we can then find all π within a particular conjugacy class. Thus, when n divides b , finding all permutiples with the same digits as a known example becomes considerably easier.

Example Using our results, we shall find all possible 5-digit $(2, 6, \sigma)$ -permutiples starting from a single example. We shall start with $(2, 6, \rho)$ -permutiple $(d_4, d_3, d_2, d_1, d_0)_6 = (4, 3, 5, 1, 2)_6 = 2 \cdot (2, 1, 5, 3, 4)_6$ with carries $(c_4, c_3, c_2, c_1, c_0) = (0, 1, 1, 1, 0)$. The reader will note that this example is a $(2, 6)$ -palintiple.

By Corollary 13 and Theorem 15, any suitable base permutation β necessarily satisfies $2d_{\rho(j)} \equiv 2d_{\beta(j)} \pmod{6}$ for all $0 \leq j \leq 4$, which, since 2 divides 6, becomes $d_{\rho(j)} \equiv d_{\beta(j)} \pmod{3}$ for all $0 \leq j \leq 4$. In matrix form we have

$$\begin{bmatrix} d_{\beta(0)} \\ d_{\beta(1)} \\ d_{\beta(2)} \\ d_{\beta(3)} \\ d_{\beta(4)} \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \pmod{3}.$$

Thus, there are four candidate base permutations: $\beta_1 = \rho$, $\beta_2 = (4, 2, 0, 1, 3)$, $\beta_3 = (4, 2, 0)(1, 3)$, and $\beta_4 = (4, 0, 1, 3)$.

For $\beta_1 = \rho$, the solution corresponding to our known example, the carry vector

is $\mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. Thus, since $\xi = (1, 2)$ is the only permutation such that P_ξ fixes both \mathbf{c}

and $P_\psi \mathbf{c}$, Theorem 11 gives us the conjugacy class with base permutation ρ which is fully described by the table below.

$(d_{\pi(4)}, d_{\pi(3)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_6$	π	τ	$(c_{\pi(4)}, c_{\pi(3)}, c_{\pi(2)}, c_{\pi(1)}, c_{\pi(0)})$
$(4, 3, 5, 1, 2)_6$	ε	ρ	$(0, 1, 1, 1, 0)$
$(4, 3, 1, 5, 2)_6$	$(1, 2)$	$(1, 2)\rho(1, 2)$	$(0, 1, 1, 1, 0)$
$(3, 5, 1, 2, 4)_6$	ψ^4	$\psi^{-4}\rho\psi^4$	$(1, 1, 1, 0, 0)$
$(3, 1, 5, 2, 4)_6$	$(1, 2)\psi^4$	$\psi^{-4}(1, 2)\rho(1, 2)\psi^4$	$(1, 1, 1, 0, 0)$

We note that Corollaries 7 and 8 give us the same results in similar fashion to our first two examples.

We now consider the conjugacy class for $\beta_2 = (4, 2, 0, 1, 3)$. We shall find this class by finding a bijection from the above class with base permutation ρ as per Theorem 14. But first we must find an example from the conjugacy class with base permutation β_2 . Provided a suitable permutation α exists, the bijection guaranteed by Theorem 14 maps permutiples with carry vector \mathbf{c} to other permutiples with the same carry vector. Therefore, if such an α exists, we know that our known example (d_k, \dots, d_0) will map to $(d_{\alpha(k)}, d_{\alpha(k-1)}, \dots, d_{\alpha(0)})_b$ so that by Theorem 15, α fixes the carry vector $(0, 1, 1, 1, 0)$ of our known $(2, 6, \rho)$ example. Thus α must contain a factor either the identity or $(4, 0)$, and a factor of either $(3, 2, 1)$, $(1, 2, 3)$, $(3, 1, 2)$, $(1, 2)$, $(1, 3)$, or $(2, 3)$. Checking the possibilities which

are not already listed in the above class by simple base-6 arithmetic, we see that either $\alpha = (4, 0)(3, 2, 1) = (1, 2)\rho$ or $\alpha = (4, 0)(3, 2) = (1, 2)\rho(1, 2)$. These values respectively give us the permutiples $(2, 5, 1, 3, 4)_6 = 2 \cdot (1, 2, 3, 4, 5)_6$, for which $\tau_2 = (0, 1)\beta_2(0, 1) = (4, 2, 1, 0, 3)$, and $(2, 5, 3, 1, 4)_6 = 2 \cdot (1, 2, 4, 3, 5)_6$, for which $\tau_2 = (1, 2)(0, 1)\beta_2(0, 1)(1, 2) = (4, 1, 2, 0, 3)$. The reader may check that both values of α yield a bijection. Using the first value, that is $\alpha = (1, 2)\rho$ with $\tau_2 = (0, 1)\beta_2(0, 1) = (4, 2, 1, 0, 3)$, Theorem 14 easily gives us the rest of the permutiples in this class and may be found in the table below.

$(d_{\pi(4)}, d_{\pi(3)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_6$	π	τ	$(c_{\pi(4)}, c_{\pi(3)}, c_{\pi(2)}, c_{\pi(1)}, c_{\pi(0)})$
$(2, 5, 1, 3, 4)_6$	$(1, 2)\rho$	$\tau_2 = (4, 2, 1, 0, 3)$	$(0, 1, 1, 1, 0)$
$(2, 5, 3, 1, 4)_6$	$(1, 2)\rho(1, 2)$	$(1, 2)\tau_2(1, 2)$	$(0, 1, 1, 1, 0)$
$(5, 1, 3, 4, 2)_6$	$(1, 2)\rho\psi^4$	$\psi^{-4}\tau_2\psi^4$	$(1, 1, 1, 0, 0)$
$(5, 3, 1, 4, 2)_6$	$(1, 2)\rho(1, 2)\psi^4$	$\psi^{-4}(1, 2)\tau_2(1, 2)\psi^4$	$(1, 1, 1, 0, 0)$

Determining the conjugacy class having the base permutation $\beta_3 = (4, 2, 0)(3, 1)$, we find an example from this class. We shall attempt to find an (n, b, β_3) -permutiple $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$. Then $\pi\beta_3\pi^{-1} = \beta_3$, so that $(\pi(4), \pi(2), \pi(0))(\pi(3), \pi(1)) = (4, 2, 0)(3, 1)$. Since the first carry of any permutiple must always be zero we know that either $\pi(0)$ equals either 0 or 4. Hence, provided π commutes with β_3 , there are 8 possibilities, among which $\pi = \beta_3$ provides us with a solution.

Therefore, by Theorem 15, with $\pi = (4, 2, 0)(1, 3)$ we have that $(c_{\pi(4)}, c_{\pi(3)}, c_{\pi(2)}, c_{\pi(1)}, c_{\pi(0)}) =$

$(1, 1, 0, 1, 0)$. So the new initial carry vector $\hat{\mathbf{c}}$ is given by $\hat{\mathbf{c}} = P_{\pi_1}\mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Now

$(0, 2)$ and $(1, 4)$ are both permutations whose corresponding matrix simultaneously fixes both $\hat{\mathbf{c}}$ and $P_{\psi}\hat{\mathbf{c}}$. Thus, by Theorem 11, we may compute the entire conjugacy class which is represented by the table below.

$(d_{\pi(4)}, d_{\pi(3)}, d_{\pi(2)}, d_{\pi(1)}, d_{\pi(0)})_6$	π	τ	$(c_{\pi(4)}, c_{\pi(3)}, c_{\pi(2)}, c_{\pi(1)}, c_{\pi(0)})$
$(5, 1, 2, 3, 4)_6$	$\pi_1 = \beta_3 = (4, 2, 0)(1, 3)$	$\pi_1^{-1}\beta_3\pi_1$	$(1, 1, 0, 1, 0)$
$(5, 1, 4, 3, 2)_6$	$\pi_2 = \pi_1(0, 2)$	$\pi_2^{-1}\beta_3\pi_2$	$(1, 1, 0, 1, 0)$
$(3, 1, 2, 5, 4)_6$	$\pi_3 = \pi_1(1, 4)$	$\pi_3^{-1}\beta_3\pi_3$	$(1, 1, 0, 1, 0)$
$(3, 1, 4, 5, 2)_6$	$\pi_4 = \pi_1(0, 2)(1, 4)$	$\pi_4^{-1}\beta_3\pi_4$	$(1, 1, 0, 1, 0)$
$(3, 4, 5, 1, 2)_6$	$\pi_5 = \pi_1\psi^2$	$\pi_5^{-1}\beta_3\pi_5$	$(1, 0, 1, 1, 0)$
$(3, 2, 5, 1, 4)_6$	$\pi_6 = \pi_1(0, 2)\psi^2$	$\pi_6^{-1}\beta_3\pi_6$	$(1, 0, 1, 1, 0)$
$(5, 4, 3, 1, 2)_6$	$\pi_7 = \pi_1(1, 4)\psi^2$	$\pi_7^{-1}\beta_3\pi_7$	$(1, 0, 1, 1, 0)$
$(5, 2, 3, 1, 4)_6$	$\pi_8 = \pi_1(0, 2)(1, 4)\psi^2$	$\pi_8^{-1}\beta_3\pi_8$	$(1, 0, 1, 1, 0)$

We now consider the final candidate $\beta_4 = (4, 0, 1, 3)$. By Theorem 15 any suitable

π in this class must satisfy $P_\pi(2P_{\beta_4} - I)\mathbf{d} = (2P_{\pi^{-1}\beta_4\pi} - I)P_\pi\mathbf{d} = (6P_\psi - I)P_\pi\mathbf{c}$, or

$$P_\pi(2P_{\beta_4} - I) \begin{bmatrix} 2 \\ 1 \\ 5 \\ 3 \\ 4 \end{bmatrix} = (6P_\psi - I)P_\pi \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

which simplifies to

$$P_\pi \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = P_\psi P_\pi \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We plainly see that P_ψ must fix the column vector $P_\pi \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. Thus, since there is no permutation π which makes this statement true, there is no π for which $(d_{\pi(k)}, d_{\pi(k-1)}, \dots, d_{\pi(0)})_b$ is an $(2, 6, \pi^{-1}\beta_4\pi)$ -permutiple.

4. Future Work and Concluding Remarks

While we have developed methods for finding all permutiples having the same digits and carries as a known example (which, as we have seen, allow us to find all desired examples when n divides b), the next obvious step forward is to find methods which give us all the desired examples.

Aside from the above, the work we have done here leaves many questions. Are there other conditions or divisibility criteria which, using the methods developed here or some slight variation thereof, allow us to find all permutiples having the same digits as a single permutiple example already in hand? How do the orders of base permutations associated with each conjugacy class relate to the order of σ ? Regarding the general permutiple problem, other questions certainly abound: what kinds of permutations can σ be? Beside Theorem 5, what sort of restrictions might there be on the order of σ ?

Another feature worth mentioning is the matrices used to represent the problem. The matrices $nP_\sigma - I$ and $bP_\psi - I$ both have interesting properties as well as their respective inverses $\frac{1}{n^{|\sigma|-1}} \sum_{\ell=0}^{|\sigma|-1} (nP_\sigma)^\ell$ and $\frac{1}{b^{k+1}-1} \sum_{\ell=0}^k (bP_\psi)^\ell$. For instance both $bP_\psi - I$ and its inverse are circulant matrices. Moreover, both $nP_\sigma - I$ and $bP_\psi - I$ have the properties of every column and row of having a respective sum of $n - 1$ and

$b - 1$. In addition to having these properties, we ask if these matrices are endowed with other special properties when (n, b, σ) -permutiples exist.

Another possible avenue of investigation is finding a way to generalize Sloane's [9] Young graph representation of palintiple structure in order to visualize permutiple structure. Such a construction would allow us to classify, and thus better understand, general permutiple structure in addition to providing a bounty of results as the Young graph has for palintiples. We also imagine that the graph would be substantially more complex. Finally, as Young graphs themselves are an interesting area of study in their own right, we conjecture that its generalization would also justify future study.

With the palintiple problem firmly in mind, the difficulty of particular digit permutation problems might make the general permutiple problem seem intractable. However, the methods developed here seem to prove otherwise; there is certainly structure that one can take advantage of. Moreover, as noted in [4], studying the general problem may very well offer insight into particular problems which study a only one kind of permutation. In particular, it might be possible to derive (in the manner described in [4]) entire palintiple classes from certain permutiple types such as those mentioned in [4].

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